

ON THE ASYMPTOTIC TENSOR NORM

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ABSTRACT. We introduce a new asymptotic one-sided and symmetric tensor norm, the latter of which can be considered as the minimal tensor norm on the category of separable C^* -algebras with homotopy classes of asymptotic homomorphisms as morphisms. We show that the one-sided asymptotic tensor norm differs in general from both the minimal and the maximal tensor norms and discuss its relation to semi-invertibility of C^* -extensions.

1. INTRODUCTION. INVERTIBILITY OF EXTENSIONS

One of the reasons, why the Brown–Douglas–Fillmore theory [2] gives so nice a classification for extensions of nuclear C^* -algebras is their invertibility. Beyond the nuclear case not much is known about general classification of C^* -extensions, but more and more examples of non-invertible extensions are coming up [1, 6, 12, 11, 5]. In [9, 10] it was suggested to weaken the notion of triviality for extensions so that more extensions would become invertible in this new sense. As it was shown by S. Wassermann, one of the reasons for non-invertibility is non-exactness and his idea relates many of examples of non-invertible extensions to the problem of coincidence for certain tensor product norms. In the present paper we develop this idea to define the asymptotic tensor norm and study this norm in hope to learn more about extensions.

Let

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0 \quad (1)$$

be an extension of A by B , i.e. a short exact sequence of C^* -algebras. We always assume that B is stable, i.e. $B \otimes \mathcal{K} \cong B$, where \mathcal{K} is the C^* -algebra of compact operators. We also assume all C^* -algebras to be separable with some obvious exceptions like the algebra $L(H)$ of bounded operators on a Hilbert space H . The extension (1) is called *split* if there is a $*$ -homomorphism $s : A \rightarrow E$ that is the right inverse for the surjection p , i.e. $p \circ s = \text{id}_A$. Among properties of split extensions one should mention their homotopy triviality (i.e. homotopy triviality of their Busby invariant). Due to stability of B , one can fix an isomorphism $M_2(B) \cong B$, which makes it possible to define a direct sum of two extensions of A by B . An extension of A by B is called *invertible* if there is another extension of A by B , such that their direct sum is split. Remark that, due to the Stinespring theorem, this is equivalent to existence of a completely positive splitting for p . Wassermann's idea of producing non-invertible extensions works as follows. Take a non-exact C^* -algebra D and an extension of the form (1) such that the sequence

$$0 \rightarrow B \otimes_{\min} D \rightarrow E \otimes_{\min} D \rightarrow A \otimes_{\min} D \rightarrow 0$$

is not exact and denote by $A \otimes_E D$ the completion of the algebraic tensor product $A \odot D$ given by the quotient $E \otimes_{\min} D / B \otimes_{\min} D$. By $\|\cdot\|_E$ we denote the corresponding tensor norm on $A \otimes_E D$.

Theorem 1.1 (S. Wassermann, [13]). *If the norm $\|\cdot\|_E$ on $A \odot D$ differs from the minimal tensor norm then the extension (1) is not invertible.*

Remark that for any commutative C^* -algebra $C_0(X)$ one can form an extension

$$0 \rightarrow C_0(X) \otimes B \rightarrow C_0(X) \otimes E \rightarrow C_0(X) \otimes A \rightarrow 0$$

out of the extension (1). It is easy to see that, if $\|\cdot\|_E \neq \|\cdot\|_{\min}$ on $A \odot D$, then $\|\cdot\|_{C_0(X) \otimes E} \neq \|\cdot\|_{\min}$ on $C_0(X) \otimes A \odot D$. Taking $X = [0, 1]$, one obtains plenty of examples of non-invertible extensions of *contractible* C^* -algebras. The first such example was found in [6] (in fact, Kirchberg's example is much more interesting than these ones because there one has $B = \mathcal{K}$).

In order to have more invertible extensions one has to change definitions. Such a change was suggested in [9, 10]:

Definition 1.2. An extension (1) is *asymptotically split* if there exists an asymptotic homomorphism $\sigma_t : A \rightarrow E$ that is the right inverse for the surjection p for every t , i.e. $p \circ \sigma_t = \text{id}_A$. An extension (1) is *semi-invertible* if there is an extension $0 \rightarrow B \rightarrow E' \rightarrow A \rightarrow 0$, such that their direct sum is asymptotically split.

It is still easy to see that asymptotically split extensions are homotopy trivial. But here we also have an almost inverse statement:

Theorem 1.3 ([10]). *Any extension of a contractible C^* -algebra is asymptotically split.*

Thus some examples of non-invertible extensions turn out to be semi-invertible.

2. ASYMPTOTIC TENSOR NORMS

Let $L(H)$ denote the algebra of bounded operators on a separable Hilbert space H . By an *asymptotic representation* of a C^* -algebra A on H we mean an asymptotic homomorphism $\mu = (\mu_t)_{t \in [0, \infty)} : A \rightarrow L(H)$. Note that a genuine representation of A on H can be considered as an asymptotic representation of A in the obvious way. By using asymptotic representations instead of genuine representations we shall now introduce two new tensor norms on the algebraic tensor product $A \odot D$ of two C^* -algebras A and D .

Let H_1, H_2 be separable Hilbert spaces and $\mu = (\mu_t)_{t \in [0, \infty)} : A \rightarrow L(H_1), \nu = (\nu_t)_{t \in [0, \infty)} : D \rightarrow L(H_2)$ two asymptotic representations. For each $a \in A$ and $d \in D$, we can define elements $a^{\mu \odot \nu}, d^{\mu \odot \nu} \in C_b([0, \infty), L(H_1 \otimes H_2))$ by

$$a^{\mu \odot \nu}(t) = \mu_t(a) \otimes 1_{H_2} \quad \text{and} \quad d^{\mu \odot \nu}(t) = 1_{H_1} \otimes \nu_t(d).$$

We can then define a $*$ -homomorphism

$$\mu \odot \nu : A \odot D \rightarrow C_b([0, \infty), L(H_1 \otimes H_2)) / C_0([0, \infty), L(H_1 \otimes H_2))$$

such that

$$\mu \odot \nu \left(\sum_i a_i \odot d_i \right) = \sum_i a_i^{\mu \odot \nu} d_i^{\mu \odot \nu}.$$

We can then define a norm $\|\cdot\|_\sigma$ (the *symmetric asymptotic tensor norm*) on $A \odot D$ by

$$\left\| \sum_i a_i \odot d_i \right\|_\sigma = \sup_{\mu, \nu} \left\| \mu \odot \nu \left(\sum_i a_i \odot d_i \right) \right\|,$$

where we take the supremum over all pairs (μ, ν) , where μ and ν are asymptotic representations of A and D , respectively. We define also a norm $\|\cdot\|_\lambda$ (the *left asymptotic tensor norm*) on $A \odot D$ by

$$\left\| \sum_i a_i \odot d_i \right\|_\lambda = \sup_{\mu, \nu} \left\| \mu \odot \nu \left(\sum_i a_i \odot d_i \right) \right\|,$$

where we take the supremum over all pairs (μ, ν) , where μ is an asymptotic representation of A and ν is a genuine representation of D . Clearly, both asymptotic tensor norms, $\|\cdot\|_\lambda$ and $\|\cdot\|_\sigma$ are cross-norms and

$$\|\cdot\|_{\min} \leq \|\cdot\|_\lambda \leq \|\cdot\|_\sigma \leq \|\cdot\|_{\max}.$$

We denote by $A \otimes_\lambda D$ and $A \otimes_\sigma D$ the C^* -algebras obtained by completing $A \odot D$ with respect to the norm $\|\cdot\|_\lambda$ and $\|\cdot\|_\sigma$ respectively.

Lemma 2.1. *Given C^* -algebras A and D , there is an asymptotic representation μ on a non-separable Hilbert space of A , such that $\|\cdot\|_\lambda = \|\mu \odot \phi(\cdot)\|$, where ϕ is a universal representation of D .*

Proof. Since any asymptotic representation is asymptotically equivalent to an equicontinuous one, so in the definition of the norm $\|\cdot\|_\lambda$ one can use only equicontinuous asymptotic representations. Let $\{\pi^\alpha\}_{\alpha \in \mathcal{A}}$ be the set of all equicontinuous asymptotic representations of A . We would like to take direct sums, but this can be possible only after reparametrization. Notice that the norm $\|\nu \odot \phi(\sum_i a_i \odot d_i)\|$ does not change after a reparametrization of an asymptotic homomorphism ν . Let $F_n \subset A$ be a sequence of finite subsets with dense union. Fix a decreasing sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ of positive numbers vanishing at infinity. For each α let $r_\alpha(t)$ be a reparametrization such that $\|\pi_{r_\alpha(t)}^\alpha(ab) - \pi_{r_\alpha(t)}^\alpha(a)\pi_{r_\alpha(t)}^\alpha(b)\| < \varepsilon_n$ for all $t \geq n$ when $a, b \in F_n$ and similar conditions for other asymptotically algebraic relations hold. Then $\mu_t = \bigoplus_{\alpha \in \mathcal{A}} \pi_{r_\alpha(t)}^\alpha$ is an asymptotic representation of A and can be used in calculation of the norm $\|\cdot\|_\lambda$.

Lemma 2.2. *Let $\phi = (\phi_t)_{t \in [0, \infty)} : A_1 \rightarrow A_2$ and $\psi = (\psi_t)_{t \in [0, \infty)} : D_1 \rightarrow D_2$ be asymptotic homomorphisms. Then their tensor product $\phi_t \otimes \psi_t$ extends to an asymptotic homomorphism from $A_1 \otimes_\sigma D_1$ to $A_2 \otimes_\sigma D_2$. If $\psi : D_1 \rightarrow D_2$ is a genuine $*$ -homomorphism then the tensor product $\phi_t \otimes \psi$ extends to an asymptotic homomorphism from $A_1 \otimes_\lambda D_1$ to $A_2 \otimes_\lambda D_2$.*

Proof. Obvious.

Proposition 2.3. *For an extension (1), suppose that there exists a C^* -algebra D and an element $x \in A \odot D$ such that $\|x\|_E > \|x\|_\lambda$. Then the extension is not semi-invertible.*

Proof. The idea of the proof is borrowed from [11]. Suppose the contrary, i.e. that (1) is semi-invertible. Then there exists an extension $0 \longrightarrow B \longrightarrow E' \xrightarrow{p'} A \longrightarrow 0$ and an asymptotic splitting $s = (s_t)_{t \in [0, \infty)} : A \rightarrow C$, where $C \subset M_2(M(B))$ is the C^* -subalgebra of the form

$$C = \left\{ \begin{pmatrix} e & b_1 \\ b_2 & e' \end{pmatrix} : b_1, b_2 \in B, e \in E, e' \in E', p(e) = p'(e') \right\}.$$

Then we have a well-defined asymptotic homomorphism $s_t \otimes_\lambda \text{id}_D : A \otimes_\lambda D \rightarrow C \otimes_{\min} D$. Let $d : C \rightarrow E$ be a completely positive contraction given by $d \begin{pmatrix} e & b_1 \\ b_2 & e' \end{pmatrix} = e$. Then the map $d \otimes \text{id}_D : C \otimes_{\min} D \rightarrow E \otimes_{\min} D$ is a well-defined contraction. Let $p_D : E \otimes_{\min} D \rightarrow A \otimes_E D$ be the quotient map. Then

$$(p_D \circ (d \otimes \text{id}_D) \circ (s_t \otimes_\lambda \text{id}_D))x = x$$

for any $x \in A \odot D$. The maps p_D and $d \otimes \text{id}_D$ are contractions and the map $s_t \otimes_\lambda \text{id}_D$ is asymptotically a contraction, hence the composition of these three maps, $A \otimes_\lambda D \rightarrow A \otimes_E D$ is asymptotically a contraction. But that contradicts $\|x\|_E > \|x\|_\lambda$.

3. DISTINCTNESS OF THE ASYMPTOTIC NORM FORM THE MINIMAL AND THE MAXIMAL ONES

Proposition 3.1. *There exist separable C^* -algebras A and D such that the norms $\|\cdot\|_\lambda$ and $\|\cdot\|_{\min}$ on $A \odot D$ are different.*

Proof. Suppose the contrary, i.e. that for any A and D these two norms coincide. Take a non-exact separable C^* -algebra D and an extension (1), with A contractible, such that the sequence $0 \rightarrow B \otimes_{\min} D \rightarrow E \otimes_{\min} D \rightarrow A \otimes_{\min} D \rightarrow 0$ is not exact. Then by Proposition 2.3 the extension is not semi-invertible, making a contradiction to Theorem 1.3.

Since $\|\cdot\|_\lambda \leq \|\cdot\|_\sigma$, the latter norm also differs from the minimal tensor norm in general.

We know nothing about associativity of the asymptotic tensor norms, but the following argument shows that either they are not associative or they look much more like the maximal tensor norm than the minimal one, at least in some group algebra examples.

Let G be an infinite hyperbolic property T group and let $p \in C^*(G)$ be the projection corresponding to the trivial representation of G . Let $\Delta : \mathbb{C}(G) \rightarrow C^*(G) \odot C^*(G)$ denote the diagonal map given by $g \mapsto g \odot g$, $g \in G$. We use the same notation Δ for $*$ -homomorphisms from $C^*(G)$ to $C^*(G) \otimes_{\max} C^*(G)$ and its quotients (like $C_r^*(G) \otimes_{\min} C^*(G)$) as well.

Corollary 3.2. *Suppose that the asymptotic tensor product is associative with respect to commutative C^* -algebras, i.e. $C_0(X) \otimes (A \otimes_\lambda D) = (C_0(X) \otimes A) \otimes_\lambda D$. Then $\Delta(p)$ in $C_r^*(G) \otimes_\lambda C^*(G)$ is not zero.*

Proof. Put $A = C_r^*(G)$, $E = C^*(G)$ and consider the extension $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$, where B is the kernel of the natural surjection of the full group C^* -algebra onto the reduced one. After tensoring this extension by $C_0[0, 1)$ one gets a semi-invertible (even an asymptotically split) extension, hence by Proposition 2.3 one has $\|\cdot\|_\lambda \geq \|\cdot\|_{C_0[0, 1) \otimes E}$ on $C_0[0, 1) \otimes C_r^*(G) \odot C^*(G)$. Suppose that $\Delta(p) = 0$ in $C_r^*(G) \otimes_\lambda C^*(G)$. Then $\|f \otimes \Delta(p)\|_\lambda = 0$ for all $f \in C_0[0, 1)$. But the quotient tensor norm is associative, so $\|f \otimes \Delta(p)\|_{C_0[0, 1) \otimes E} = \|f\| \cdot \|\Delta(p)\|_E = \|f\|$, since $\Delta(p) \neq 0$ in $C_r^*(G) \otimes_E C^*(G)$ (cf. Lemma 6.2.9 of [4]).

Similarly, associativity of the symmetric asymptotic tensor product implies that $\Delta(p)$ is not zero in $C_r^*(G) \otimes_\sigma C_r^*(G)$.

Recall that $\Delta(p)$ is not zero in $C_r^*(G) \otimes_{\max} C_{(r)}^*(G)$ and it is zero in $C_r^*(G) \otimes_{\min} C_{(r)}^*(G)$ [4].

Proposition 3.3. *Let D be a C^* -algebra without the weak expectation property of Lance [7] and let F_∞ denote a free group on an infinite set of generators. Then $C^*(F_\infty) \otimes_{\max} D \neq C^*(F_\infty) \otimes_\lambda D = C^*(F_\infty) \otimes_{\min} D$.*

Proof. Recall that D has the weak expectation property if $D \otimes_{\max} E \subset C \otimes_{\max} E$ canonically for any C^* -algebra E and for any C^* -algebra C containing D as C^* -subalgebra. By Proposition 1.1 of [6] one has $C^*(F_\infty) \otimes_{\max} D \neq C^*(F_\infty) \otimes_{\min} D$, so we have to check that the asymptotic norm coincides with the minimal one. Since both norms are cross-norms, it is sufficient to check that they coincide on a dense set, e.g. on finite sums $c = \sum_i \gamma_i \odot d_i \in \mathbb{C}[F_\infty] \odot D$, where all $\gamma_i \in F_\infty$ are words on a finite number of generators, g_1, \dots, g_n , of F_∞ . For any

asymptotic representation $\rho_t : F_\infty \rightarrow L(H)$ without loss of generality we can assume that for big enough t the operators $\rho_t(g_k)$, $k = 1, \dots, n$, are unitaries. Since F_∞ is free, this means that in calculation of the asymptotic norm for c we can use only genuine representations π_t of F_∞ given by $\pi_t(g_k) = \begin{cases} \rho_t(g_k), & \text{if } k \leq n, \\ 1, & \text{if } k > n \end{cases}$ instead of asymptotic ones, hence the asymptotic norm coincides with the minimal one.

As an example of a C^* -algebra without the weak expectation property one can use the reduced group C^* -algebra of $SL_2(\mathbb{Z})$ [6].

Thus we have established the following result.

Theorem 3.4. *The tensor norm $\|\cdot\|_\lambda$ differs both from the minimal and the maximal tensor norms. The tensor norm $\|\cdot\|_\sigma$ differs from the minimal tensor norm.*

We don't know if the symmetric asymptotic tensor norm differs from the maximal one, although their coincidence seems very unlikely. The similar argument does not work, since it reduces to a long-standing problem if $C^*(F_\infty) \otimes_{\max} C^*(F_\infty) \neq C^*(F_\infty) \otimes_{\min} C^*(F_\infty)$ [6].

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